

List version of $(p,1)$ -total labellings *

Yong Yu, Guanghui Wang, Guizhen Liu [†]

School of Mathematics, Shandong University, Jinan 250100, P.R.China

Abstract

The $(p,1)$ -total number $\lambda_p^T(G)$ of a graph G is the width of the smallest range of integers that suffices to label the vertices and the edges of G such that no two adjacent vertices have the same label, no two incident edges have the same label and the difference between the labels of a vertex and its incident edges is at least p . In this paper we consider the list version. Let $L(x)$ be a list of possible colors for all $x \in V(G) \cup E(G)$. Define $C_{p,1}^T(G)$ to be the smallest integer k such that for every list assignment with $|L(x)| = k$ for all $x \in V(G) \cup E(G)$, G has a $(p,1)$ -total labelling c such that $c(x) \in L(x)$ for all $x \in V(G) \cup E(G)$. We call $C_{p,1}^T(G)$ the $(p,1)$ -total labelling choosability and G is list L -($p,1$)-total labelable.

In this paper, we present a conjecture on the upper bound of $C_{p,1}^T$. Furthermore, we study this parameter for paths and trees in Section 2. We also prove that $C_{p,1}^T(K_{1,n}) \leq n + 2p - 1$ for star $K_{1,n}$ with $p \geq 2, n \geq 3$ in Section 3 and $C_{p,1}^T(G) \leq \Delta + 2p - 1$ for outerplanar graph with $\Delta \geq p + 3$ in Section 4.

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1 Introduction

In this paper, the term *graph* is used to denote a simple connected graph G with a finite vertex set $V(G)$ and a finite edge set $E(G)$. The degree of a vertex v in G is the number of edges incident with v and denoted by $d_G(v)$. We write $\delta(G) = \min\{d_G(v) : v \in V(G)\}$ and $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$ to denote the minimum degree and maximum degree of G , respectively. We sometimes write $V, E, d(v), \Delta, \delta$ instead of $V(G), E(G), d_G(v), \Delta(G), \delta(G)$, respectively. A function L is called an *assignment* for a graph G if it assigns a list $L(x)$ of possible labels (or colors) to each element $x \in V(G) \cup E(G)$. A *k-assignment* is a list assignment where all lists have the same cardinality k , that is, $|L(x)| = k$ for all $x \in V(G) \cup E(G)$. We shall assume throughout that the labels (or colors) are

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[†]Corresponding author. E-mail address: gzliu@sdu.edu.cn.

natural numbers. Our terminology and notation will be standard except where indicated. Readers are referred to [2] for undefined terms.

Let p be a nonnegative integer. A k -($p, 1$)-total labelling of a graph G is a function c from $V(G) \cup E(G)$ to the color set $\{0, 1, \dots, k\}$ such that $c(u) \neq c(v)$ if $uv \in E(G)$, $c(e) \neq c(e')$ if e and e' are two adjacent edges, and $|c(u) - c(e)| \geq p$ if vertex u is incident to the edge e . The minimum k such that G has a k -($p, 1$)-total labelling is called the ($p, 1$)-total labelling number and denoted by $\lambda_p^T(G)$. Let us denote by $\chi_{p,1}^T(G)$ the minimum number of colors(labels) needed for an ordinary ($p, 1$)-total labelling for describing conveniently in this paper. Obviously, we have $\chi_{p,1}^T(G) = \lambda_p^T(G) + 1$. When $p = 1$, the ($1, 1$)-total labelling is the well-known total coloring of graphs, and $\chi_{1,1}^T(G) = \chi''(G)$ where $\chi''(G)$ denotes the total chromatic number.

Here we present the concept list ($p, 1$)-total labelling. Suppose L is an assignment for a graph G . If G has a ($p, 1$)-total labelling c such that $c(x) \in L(x)$ for all $x \in V(G) \cup E(G)$, then we say that c is an L -($p, 1$)-total labelling of G , and G is L -($p, 1$)-total labelable. Furthermore, if G is L -($p, 1$)-total labelable for any L with $|L(x)| = k$ for each $x \in V(G) \cup E(G)$, we say that G is k -($p, 1$)-total choosable. The ($p, 1$)-total labelling choosability, denoted by $C_{p,1}^T(G)$, is the minimum k such that G is k -($p, 1$)-total choosable.

Obviously, this concept is a common generalization of list colorings and ($p, 1$)-total labellings. The ($p, 1$)-total labelling of graphs was introduced by Havet and Yu [6]. It was shown that $\lambda_p^T(G) \leq 2\Delta + p - 1$ for any graph G , and if $\Delta \geq 3$ then $\lambda_p^T(G) \leq 2\Delta$, if $\Delta \geq 5$ is odd then $\lambda_p^T(G) \leq 2\Delta - 1$. The special cases for $p = 2$ were also investigated in this paper. Some kind of special graphs have also been studied, e.g., complete bipartite graphs for $p = 2$ [9], planar graphs [1], trees for $p = 2$ [12], graphs with a given maximum average degree [10], complete graphs [6], etc. In [6], Havet and Yu gave a conjecture that $\lambda_p^T(G) \leq \Delta + 2p - 1$ for any graph G , which extends the well known Total Coloring Conjecture in which $p = 1$.

The *incidence graph* of a graph G , denoted by $S_I(G)$, is the graph obtained from G by replaced each edge by a path of length 2. Motivated by the Frequency Channel Assignment Problem, Griggs and Yeh [5] first introduced the $L(2, 1)$ -labelling of graphs. This notion was subsequently extended to a general form, named as $L(p, q)$ -labelling of graphs. The $L(p, q)$ -labelling, especially the $L(2, 1)$ -labelling, of graphs have been studied rather extensively in recent years. Kohl et al. [4] investigated the list version of $L(p, q)$ -labellings and obtained some interesting results. As mentioned in [6], the $L(p, 1)$ -labelling of $S_I(G)$ is equivalent to the ($p, 1$)-total labelling of graph G . We still noticed that the ($p, 1$)-total labelling is a special case of an $[r, s, t]$ -coloring of graphs with $r = s = 1, t = p$, which was introduced in [7]. Hence it is easy to see :

Observation 1. Let G be a graph. Then

$$\chi_l^{p,1}(S_I(G)) = C_{p,1}^T(G) = \chi_l^{1,1,p}(G),$$

where $\chi_l^{p,1}(G)$ and $\chi_l^{1,1,p}(G)$ denote the minimum number k such that G is k - $L(p, 1)$ -labelling choosable and k - $[1, 1, p]$ -coloring choosable, respectively.

In Section 2, we give some general bounds for $C_{p,1}^T$ for paths and trees. After that, we present a conjecture on the upper bound of $C_{p,1}^T$ for any graph G .

In Section 3, we show that $C_{p,1}^T(K_{1,n}) \leq n + 2p - 1$ where $p \geq 2, n \geq 3$.

In Section 4, we discuss the value for C_p^T for outerplanar graphs. We prove that $C_{p,1}^T(G) \leq \Delta + 2p - 1$ for all outerplanar graph G with $\Delta \geq p + 3$, and we conjecture that the upper bound is still true without the maximum degree restriction.

2 Basic results on $C_{p,1}^T$

At first, by using Observation 1 we try to give some bounds for paths and trees. Then we give a conjecture on the upper bound for any graph G .

Lemma 2.1 ([8] or [4]) *Let P_k be a path with k vertices. Then*

$$\chi^{p,1}(P_k) = \begin{cases} p + 1, & k = 2; \\ p + 2, & k = 3, 4; \\ p + 3, & k \geq 5. \end{cases}$$

Theorem 2.2 *Let P_k be a path with k vertices. Then*

$$\chi_{p,1}^T(P_k) = \begin{cases} p + 2, & k = 2; \\ p + 3, & k \geq 3. \end{cases}$$

Proof. Let $S_I(P_k)$ be the incidence graph of P_k , then $S_I(P_k) = P_{k'}$ is still a path with $k' = 2k - 1$. By Observation 1 and Lemma 2.1, we have $\chi_{p,1}^T(P_2) = \chi^{p,1}(P_3) = p + 2$, and when $k \geq 3$ we have $\chi_{p,1}^T(P_k) = \chi^{p,1}(P_{k'}) = p + 3$ since $k' = 2k - 1 \geq 5$. ■

Lemma 2.3 ([8]) *Let $P_k = v_1 \cdots v_k$ be a path and $k > 2p$. Then we have $2p \leq \chi_l^{p,1}(P_k) \leq 2p + 1$.*

Theorem 2.4 *Let $P_k = v_1 \cdots v_k$ be a path. Then $C_{p,1}^T(P_k) \leq 2p + 1$. Moreover, if $k > p$, then we have $2p \leq C_{p,1}^T(P_k) \leq 2p + 1$.*

Proof. $C_{p,1}^T(P_k) \leq 2p + 1$ is obvious since we can color the vertices and edges of the path sequentially in its order by a greedy algorithm. When $k > p$, an analogous argument with the proof in Theorem 2.2 shows that $C_{p,1}^T(P_k) = \chi_l^{p,1}(S_I(P_k))$. Then by Lemma 2.3 we have $2p \leq C_{p,1}^T(P_k) \leq 2p + 1$. ■

When $p = 2$, we have $C_{2,1}^T(P_k) \leq 5$ with $k \leq 3$ by Theorem 2.4. By the definition of $C_{p,1}^T$, it is easy to see that $C_{p,1}^T(G) \geq \chi_l^{p,1}(G)$. Then $C_{2,1}^T(P_k) \geq \chi_l^{2,1}(P_k) = 5$ by Theorem 2.2. Therefore, $C_{2,1}^T(P_k) = 5 = 2p + 1$ when $k \geq 3$. So the upper bound of Theorem 2.4 is tight.

Lemma 2.5 ([8]) *For all trees T , all d and all $s \geq 1$, we have $\chi_l^{d,s}(T) \leq 2d - 1 + s\Delta$.*

Theorem 2.6 *Let T_n be a tree with n vertices. Then we have $C_{p,1}^T(T_n) \leq \Delta + 2p - 1$.*

Proof. Let $S_I(T_n)$ be the incidence graph of T_n . $S_I(T_n)$ is still a tree with $n' = 2n - 1$ vertices and $\Delta(T_{n'}) = \Delta(T_n)$. By Lemma 2.5, let $d = p, s = 1$ we have $\chi_l^{p,1}(T_{n'}) \leq \Delta + 2p - 1$. Therefore, by Observation 1 we obtain $C_{p,1}^T(T_n) = \chi_l^{p,1}(T_{n'}) \leq \Delta + 2p - 1$. ■

Lemma 2.7 ([8]) *If T is a tree with maximum degree Δ , $p \leq \Delta$, and there is a vertex $v \in V(G)$ such that v and all of its neighbors have degree Δ , then $\chi_l^{p,1}(T_n) = \Delta + 2p - 1$.*

By Lemma 2.7, if $T = P_n$, $n \geq 5$ and $p = 2$, $C_{2,1}^T(T_n) = \chi_l^{2,1}(S_I(T_n)) = \chi_l^{p,1}(T_{n'}) = \Delta(T_{n'}) + 2p - 1 = \Delta(T_n) + 2p - 1$. That is to say, the upper bound of Theorem 2.6 is also tight.

It is known to all that for list version of edge colorings and total colorings there are list edge coloring conjecture (LECC) and list total coloring conjecture (LTCC) as follows:

$$(1)\chi'_l(G) = \chi'(G); (2)\chi''_l(G) = \chi''(G).$$

Therefore, it is natural for us to conjecture that it may be also true for $(p,1)$ -total labellings. That is, $C_{p,1}^T(G) = \chi_{p,1}^T(G) (= \lambda_p^T(G) + 1)$. Unfortunately, we could find counterexamples with $C_{p,1}^T(G)$ is strictly greater than $\chi_{p,1}^T(G)$. Taking P_k with $k > p$ as an example, we have $\chi_{p,1}^T(G) \leq p + 3$ by Theorem 2.2 but $C_{p,1}^T(P_k) \geq 2p$ by Theorem 2.4, which is strictly greater than $\chi_{p,1}^T(P_k)$ when $p \geq 4$.

Although we can not present a conjecture like LECC or LTCC, we may conjecture an upper bound for $C_{p,1}^T(G)$ for any graph G :

Conjecture 2.8 *Let G be a simple graph with maximum degree Δ . Then*

$$C_{p,1}^T(G) \leq \Delta + 2p.$$

Obviously, the conjecture is true for paths and trees by Theorem 2.4 and 2.6. Havet and Yu [6] gave a similar conjecture on $\lambda_p^T(G)$. They also showed that $\lambda_p^T(K_n) = n + 2p - 2$ for complete graph with $n \geq 6p^2 - 10p + 4$ was even. Then $C_{p,1}^T(K_n) \geq \chi_{p,1}^T(K_n) = \lambda_p^T(K_n) + 1 = \Delta + 2p$. Therefore, the bound in Conjecture 2.8 is tight.

3 Stars

In this section, we prove that the conjecture above is true for stars. Actually, we can improve the bound by one for stars.

Obviously, $C_{1,1}^T(K_{1,n}) = \chi''_l(K_{1,n}) = n + 1$. When $n \leq 2$, $K_{1,n}$ is equivalent to P_{n+1} , which condition have been shown in Theorem 2.4. Therefore, we only need to consider the case when $p \geq 2$ and $n \geq 3$.

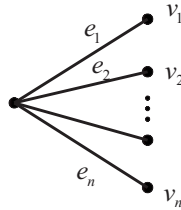


Figure 1

Theorem 3.1 *Let $K_{1,n}$ be a star with $n \geq 3$ and $p \geq 2$. Then*

$$C_{p,1}^T(K_{1,n}) \leq n + 2p - 1.$$

Proof. Assume $|L(x)| = k$ with $k = n + 2p - 1$ for all $x \in V \cup E$. Denote the maximum vertex by w and the others by v_1, \dots, v_n . Denote the edges by e_1, \dots, e_n , respectively (see Figure 1). Denote the colors $\{x - (p - 1), \dots, x - 1, x, x + 1, \dots, x + (p - 1)\}$ by $\|x\|_p$ and the labelling of $K_{1,n}$ by c . Then if we label $x \in V \cup E$ with color $\alpha \in L(x)$, we sometimes denote that by $c(x) = \alpha$.

First, label w by the minimum color α of its list and let $L'(e_j) = L(e_j) \setminus \{\|\alpha\|_p\}$, $L'(v_j) = L(v_j) \setminus \{\alpha, \|c(e_j)\|_p\}$ for all $j \in \{1, 2, \dots, n\}$. Then we have $|L'(e_j)| \geq n - 1$ and $|L'(v_j)| \geq k - (2p - 1 + 1) = n - 1 \geq 1$. Therefore, we just need to consider the coloring, denoted by c , of edges e_j for all j . The coloring of vertices v_j is obvious since $|L'(v_j)| \geq 1$. Then we get an L -($p, 1$)-total labelling of $K_{1,n}$ with the assignment L .

Assume that at least one of the lists, say $L'(e_1)$, still contains at least n colors. We give an algorithm for the edge coloring as follows:

Step 1: Let $i = 1, S = \emptyset, a_i = e_1$;

Step 2: Determine the minimum color m of the union of the lists of all uncolored edges. That is, $m = \min\{x \mid x \in \bigcup_{e_p \in E \setminus S} L'(e_p)\}$;

Step 3: If $L'(e_1)$ contains m and no other uncolored edges has m in its list, then let $e'_i = a_i$; otherwise, choose another e_k with $m \in L'(e_k)$ and let $e'_i = e_k$.

Step 4: Let $c(e'_i) = m, S = S \cup \{e'_i\}$;

Step 5: If $i = n$, then stop; otherwise, delete m from the lists of uncolored edges, that is, let $L'(e_p) = L'(e_p) \setminus \{m\}$ for all $e_p \in E \setminus S$;

Step 6: If $e'_i = a_i$, then $a_{i+1} = e_p$ where $|L'(e_p)| \geq n - i, e_p \in E \setminus S$; else $a_{i+1} = a_i$;

Step 7: $i = i + 1$, turn Step 2.

We delete at most one color in every step. So if e_1 is the last edge colored by our algorithm, then the coloring is possible since the list of e_1 has at least one color left by assumption. If e_1 is not the last edge, then the coloring of e_1 deletes no color from any list of $E \setminus S$. Suppose e_1 get colored by the i th loop for some i . Then we have deleted at most $i - 1$ colors from the list of e_p for all $e_p \in E \setminus S$, and we can choose some e_p as the new beginning of our algorithm since we have $|L'(e_p)| \geq n - i$ at the beginning of the next loop.

Thus, each edge list $L'(e_j)$ has exactly $n - 1$ colors. That means $\|\alpha\|_p \subseteq L(e_j)$ for all j . If we could not finish the coloring, then an analogous fact must hold for every color $\beta \in L(w)$. Therefore, $\{\alpha - (p - 1), \dots, \alpha - 1\} \cup L(w) \subseteq L(e_j)$. So we have $k = |L(e_j)| \geq |\{\alpha - (p - 1), \dots, \alpha - 1\} \cup L(w)| = p - 1 + k$, which is a contradiction. \blacksquare

Lemma 3.2 ([6]) *Let G be a bipartite graph. Then*

$$\Delta + p - 1 \leq \lambda_p^T(G) \leq \Delta + p.$$

Moreover, if $p \geq \Delta$ or G is regular, then $\lambda_p^T(G) = \Delta + p$.

Theorem 3.3 *Let $K_{1,n}$ be a star. Then*

$$\chi_{p,1}^T(K_{1,n}) = \begin{cases} n + p, & p < n; \\ n + p + 1, & p \geq n. \end{cases}$$

Proof. By Lemma 3.2 and $\chi_{p,1}^T = \lambda_p^T + 1$, we have $n + p \leq \chi_{p,1}^T(K_{1,n}) \leq n + p + 1$ and $\chi_{p,1}^T(K_{1,n}) = n + p + 1$ when $p \geq n$. If $p \leq n - 1$, then we give a $(p,1)$ -total labelling of $K_{1,n}$ with colors $\{1, 2, \dots, n + p\}$. Suppose $K_{1,n}$ is defined as Figure 1. We color w with $n + p$ and color e_j with j for all $j \in \{1, 2, \dots, n\}$. After that we color v_j with $p + j$ for $j = 1, \dots, n - 1$ and color v_n with color 1. Since $n - 1 \geq p$, this coloring is a proper $(p,1)$ -total labelling of $K_{1,n}$. Therefore, $\chi_{p,1}^T(K_{1,n}) = n + p$ when $p < n$. ■

When $p = 2$, we have $\chi_{2,1}^T(K_{1,n}) = n + 2$ by Theorem 3.3. Then $C_{2,1}^T(K_{1,n}) \geq \chi_{2,1}^T(K_{1,n}) = n + 2$. On the other hand, we also have $C_{2,1}^T(K_{1,n}) \leq n + 2 \times 2 - 1 = n + 2$ by Theorem 3.1. That is, $C_{2,1}^T(K_{1,n}) = n + 2 = n + 2 \times p - 1$. Therefore, the upper bound of Theorem 3.1 is tight when $p = 2$.

4 Outerplanar graphs

In this section, we discuss the $C_{p,1}^T$ of outerplanar graphs G . An outerplanar graph is a planar graph that can be drawn on the Euclidean plane such that there exists a face f with all $v \in V(G)$ belong to f . For these special graphs, we give a theorem as follows:

Theorem 4.1 *Let G be an outerplanar graph with maximum degree $\Delta \geq p + 3$. Then*

$$C_{p,1}^T(G) \leq \Delta + 2p - 1.$$

We will prove Theorem 4.1 by contradiction. Before that, we need a configuration lemma as follows:

Lemma 4.2 ([3]) *Every outerplanar graph G with $\delta(G) = 2$ contains one of the following configurations (see Figure 2):*

- (C1) *two adjacent 2-vertices u and v ;*
- (C2) *a 3-face $[uv_1v_2]$ with $d(u) = 2$ and $d(v_1) = 3$;*
- (C3) *two 3-face $[u_1v_1x]$ and $[u_2v_2x]$ such that $d(x) = 4$ and $d(u_1) = d(u_2) = 2$.*

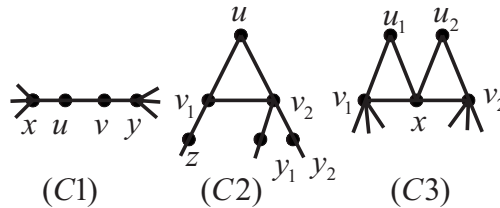


Figure 2

Proof of Theorem 4.1. Let H be a minimal counterexample in terms of $|V(G)| + |E(G)|$ to Theorem 4.1. L is the k assignment defined on $V(H) \cup E(H)$ and $k = \Delta + 2p - 1$. Denote the $(p, 1)$ -total labelling of H by c . Then if we label $x \in V(H) \cup E(H)$ with color $\alpha \in L(x)$, we sometimes denote that by $c(x) = \alpha$. Denote by $L'(x)$ the set of colors still available to color the element $x \in V(H) \cup E(H)$ such that the labelling is a proper $(p, 1)$ -total labelling. We still use $\|x\|_p$ to denote the color set $\{x - (p - 1), \dots, x - 1, x, x + 1, \dots, x + (p - 1)\}$.

Claim 1. $\delta(H) \geq 2$.

Proof. If $\delta(H) = 1$. Suppose that $e = uv \in E(H)$ and $d(v) = 1$. The graph $H' = H \setminus \{v\}$ still satisfies the demands of the theorem. By the minimality of H , H' is L -($p, 1$)-total labelable. Without loss of generality, we suppose the labelling is c . Then at most $\Delta - 1 + 2p - 1$ colors are forbidden for the labelling of edge e . So we can choose a color for e from $L'(e)$ since $|L'(e)| \geq |L(e)| - (\Delta - 1 + 2p - 1) = 1$. After that we color v from $L'(v) = L(v) \setminus \{c(u), \|c(e)\|_p\}$. It is possible since $|L'(v)| \geq k - (1 + 2p - 1) = \Delta - 1$. Then we extend the labelling c to H , which is a contradiction. ■

Therefore, $\delta(H) = 2$. By Lemma 4.2, H contains one of the configurations C1–C3. Next, we will show that in each case of C1–C3 we can obtain a labelling such that H is L -($p, 1$)-total labelable. Then we get a contradiction:

(C1) Let x be the neighbor of u different from v and y the neighbor of v different from u . Let $H' = H \setminus e$ where $e = uv$. Then H' still satisfies the demands of the theorem. By the minimality of H , H' is L -($p, 1$)-total labelable. Remove the colors of vertex u and v . After that we define a list of available colors for each of u, v and e as follows.

$$L'(u) = L(u) \setminus \{c(x), \|c(ux)\|_p\},$$

$$L'(v) = L(v) \setminus \{c(y), \|c(vy)\|_p\},$$

$$L'(e) = L(e) \setminus \{c(ux), c(vy)\}.$$

Since $|L| = k = \Delta + 2p - 1$ and $\Delta \geq p + 3$, it follows that

$$|L'(u)| \geq k - (1 + 2p - 1) \geq p + 2,$$

$$|L'(v)| \geq k - (1 + 2p - 1) \geq p + 2,$$

$$|L'(e)| \geq k - 2 \geq 3p.$$

Let $m = \min\{x \mid x \in L'(u) \cup L'(v) \cup L'(e)\}$.

Case 1. $m \notin L'(u) \cup L'(v)$. Let $c(e) = m$ and at most $p - 1$ colors are unavailable for coloring u, v . Then at least 3 colors are left in $L'(u)$ and $L'(v)$. So we can choose two left colors from the list of u, v such that $c(u) \neq c(v)$.

Case 2. $m \in L'(u)$ or $L'(v)$. Without loss of generality, say $m \in L'(u)$. Let $c(u) = m$. Then at most p colors are unavailable for coloring e and at most one color for v . Let

$$L''(v) = L'(v) \setminus \{c(u)\}, L''(e) = L'(e) \setminus \{\|c(u)\|_p\}.$$

Then we have

$$|L''(v)| \geq p + 2 - 1 \geq p + 1, |L''(e)| \geq 3p - (2p - 1) \geq p + 1.$$

Let $m_1 = \min\{x \mid x \in L''(v) \cup L''(e)\}$.

Case 2.1. $m_1 \in L''(v)$. Let $c(v) = m_1$ then we delete at most p colors from $L''(e)$. So we can color e since at least $|L''(e)| - p \geq 1$ colors are still available for e .

Case 2.2. $m_1 \notin L''(v)$. Let $c(e) = m_1$ then we delete at most $p - 1$ colors from $L''(v)$. So we can color v since at least $|L''(v)| - (p - 1) \geq 2$ colors are still available for v .

In any case, we extend the labelling c to H for (C1), which is a contradiction.

(C2) Let z be the neighbor of v_1 and v_2 . Let $H' = H \setminus uv_1$. Then H' still satisfies the demands of the theorem. By the minimality of H , H' is L -($p, 1$)-total labelable. Remove the colors of vertex u . After that we define a list of available colors for each of u and uv_1 as follows.

$$L'(u) = L(u) \setminus \{c(v_1), c(v_2), \|c(uv_2)\|_p\},$$

$$L'(uv_1) = L(uv_1) \setminus \{c(v_1z), c(v_1v_2), c(uv_2), \|c(v_1)\|_p\}.$$

Since $|L| = k = \Delta + 2p - 1$ and $\Delta \geq p + 3$, it follows that

$$|L'(u)| \geq k - (2 + 2p - 1) \geq p + 1,$$

$$|L'(uv_1)| \geq k - (3 + 2p - 1) \geq p.$$

Let $m = \min\{x \mid x \in L'(u) \cup L'(uv_1)\}$. If $m \in L'(uv_1)$, let $c(uv_1) = m$. Then at most p colors are unavailable for coloring u . So we can color u since at least $|L'(u)| - p \geq 1$ colors are still available for u ; otherwise, let $c(u) = m$ and at most $p - 1$ colors are unavailable for coloring uv_1 . So we can color uv_1 since at least $|L'(uv_1)| - (p - 1) \geq 1$ colors are still available for uv_1 . Then we extend the labelling c to H for (C2), which is a contradiction.

(C3) Let $H' = H \setminus xu_1$. By the minimality of H , H' is L -($p, 1$)-total labelable. Remove the colors of vertex u_1 . After that we define the lists of available labels for u_1 and xu_1 as follows.

$$L'(u_1) = L(u_1) \setminus \{c(v_1), c(x), \|c(u_1v_1)\|_p\},$$

$$L'(xu_1) = L(xu_1) \setminus \{c(v_1x), c(v_1u_1), c(xv_2), c(xu_2), \|c(x)\|_p\}.$$

Since $|L| = k = \Delta + 2p - 1$ and $\Delta \geq p + 3$, it follows that

$$|L'(u_1)| \geq k - (2 + 2p - 1) \geq p + 1,$$

$$|L'(xu_1)| \geq k - (4 + 2p - 1) \geq p - 1.$$

If $|L'(xu_1)| \geq p$, we can color xu_1 and u_1 as we have done in Case (C2); otherwise, we have $|L'(xu_1)| = p - 1$ and $\Delta = p + 3$. Let

$$m = \min\{x \mid x \in L'(u_1) \cup L'(xu_1)\},$$

$$M = \max\{x \mid x \in L'(u_1) \cup L'(xu_1)\},$$

$$L'_0(u_1) = L'(u_1) \setminus \{m, M\}.$$

Denote by m_1 (M_1) and a (b) the minimum (maximum) number of $L'_0(u_1)$ and $L'(xu_1)$, respectively.

If $P = xu_1$ has not a partial list L -($p,1$)-total labelling for u_1 and xu_1 , then we have some claims as follows.

Claim 2. $L'(xu_1)$ is a series of $p - 1$ successively integers. That is, $|L'(xu_1)| = p - 1$ and $b - a + 1 = p - 1$.

Proof. If $m \in L'(xu_1)$, let $c(xu_1) = m$. Then at most colors $\{m, m + 1, \dots, m + (p - 1)\}$ are forbidden for list $L'(u_1)$. Therefore, we can color u_1 with at least one color from the available colors of $L'(u_1)$. If $M \in L'(xu_1)$, then we can finish the partial list-($p,1$)-total labelling for u_1 and xu_1 with an analogous analysis. Therefore, we have $a \geq m + 1$. If $b - a + 1 \geq p$, then $b - m \geq b - (a - 1) \geq p$. We can color u_1 with m and xu_1 with b . Obviously, this is a partial list-($p,1$)-total labelling for u_1 and xu_1 , which is a contradiction to our assumption. Therefore, $b - a + 1 = p - 1$. ■

Claim 3. $L'(xu_1) = L'_0(u_1)$ and $m_1 = m + 1, M_1 = M - 1$.

Proof. If $a \leq m_1 - 1$ or $b \geq M_1 + 1$, then we have $M - a \geq (M_1 + 1) - (m_1 - 1) = (M_1 - m_1 + 1) + 1 \geq p$ or $b - m \geq (M_1 + 1) - (m_1 - 1) = (M_1 - m_1 + 1) + 1 \geq p$. We can color u_1 with m and xu_1 with b or we can color u_1 with M and xu_1 with a . Then we obtain a partial list-($p,1$)-total labelling for u_1 and xu_1 . Therefore, $a \geq m_1$ and $b \leq M_1$. If $a \geq m_1 + 1$ or $b \leq M_1 - 1$, then we also have $b - m \geq b - (m_1 - 1) \geq b - a + 2 = p$ or $M - a \geq (M_1 + 1) - a \geq (b + 1) + 1 - a = p$ since $b - a + 1 = p - 1$ by Claim 2. We still obtain a partial list-($p,1$)-total labelling for u_1 and xu_1 . Therefore, $a = m_1$ and $b = M_1$. By our assumption, $M_1 - m_1 + 1 \geq |L'_0(u_1)| = |L'(u_1)| - 2 \geq p - 1$. Thus, we have $M_1 - m_1 + 1 = |L'_0(u_1)| = p - 1$. Together with $a = m_1, b = M_1$ and Claim 2, we obtain $L'(xu_1) = L'_0(u_1)$. ■

Claim 4. $m_1 = m + 1$ and $M_1 = M - 1$.

Proof. If $m_1 \geq m + 2$ or $M_1 \leq M - 2$, then we have $b - m \geq b - (m_1 - 2) = M_1 - m_1 + 2 = p$ or $M - a \geq (M_1 + 2) - a = M_1 - m_1 + 2 = p$ by Claim 3. We can color u_1 with m and xu_1 with b or we can color u_1 with M and xu_1 with a . Then we obtain a partial list-($p,1$)-total labelling for u_1 and xu_1 , which is a contradiction to our assumption. ■

That is to say,

$$L'(u_1) = \{m, m + 1, \dots, m + (p - 1), m + p\},$$

$$L'(xu_1) = \{m + 1, \dots, m + (p - 1)\}$$

where $a = m_1 = m + 1, b = M_1 = m + (p - 1), M = m + p$.

Claim 2–4 show that if $L'(u_1), L'(xu_1)$ are not defined as above, we can obtain a partial list-($p,1$)-total labelling for u_1 and xu_1 . Next, we show that we can obtain a partial list-($p,1$)-total labelling for u_1 and xu_1 even if $L'(u_1), L'(xu_1)$ satisfies Claim 2–4 :

Since $|L'(xu_1)| = p - 1$, it follows that $\Delta = p + 3$ and $\{c(v_1x), c(v_1u_1), c(xv_2), c(xu_2), \|c(x)\|_p\}$ are all distinct. Otherwise, $|L'(xu_1)| > \Delta + 2p - 1 - (2p + 3) = p - 1$ which is a contradiction.

In particular, $c(v_1u_1) \notin \{c(v_1x), c(xv_2), c(xu_2), \|c(x)\|_p\}$. We interchange the colors of xv_1 and u_1v_1 . After that, we define a new list L'' of available colors for u_1 and xu_1 , then

$$L''(xu_1) = L'(xu_1) = L(xu_1) \setminus \{c(v_1x), c(v_1u_1), c(xv_2), c(xu_2), \|c(x)\|_p\},$$

$$L''(u_1) = L(u_1) \setminus \{c(v_1), c(x), \|c(xv_1)\|_p\}.$$

Since $c(xv_1) \neq c(u_1v_1)$, we see that $L''(u_1) \neq L'(u_1)$ and $|L''(u_1)| \geq |L'(u_1)|$. Then we have $m' \leq m - 1$ or $M' \geq M + 1$ where m', M' denote the minimum and maximum number of $L''(u_1)$. Otherwise, we have $m' \geq m$, $M' \leq M$ and $|L''(u_1)| \leq M' - m' + 1 \leq M - m + 1 = |L'(u_1)|$. Since $|L''(u_1)| \geq |L'(u_1)|$, $|L''(u_1)| = |L'(u_1)|$ and $m' = m$, $M' = M$. That is, $L''(u_1) = L'(u_1)$, which is a contradiction.

If $m' \leq m - 1$ then we have $m + (p - 1) - m' \geq m + (p - 1) - (m - 1) = p$. Let $c(u_1) = m'$ and $c(xu_1) = m + (p - 1)$. Then we obtain a partial list- $(p, 1)$ -total labelling for u_1 and xu_1 . If $M' \geq M + 1$, then we have $M' - a = M' - (m + 1) \geq (M + 1) - (m + 1) = (p + 1) - 1 = p$. Let $c(u_1) = M'$ and $c(xu_1) = a = m + 1$. Then we obtain a partial list- $(p, 1)$ -total labelling for u_1 and xu_1 .

Any way, we extend the labelling c to H for (C3), which is a contradiction. ■

Then we complete the proof of Theorem 4.1. ■

When $p = 2$, our result generalized a result of Chen and Wang [3]:

Corollary 4.3 ([3]Theorem 7) *If G is an outerplanar graph with $\Delta(G) \geq 5$, then $\lambda_2^T(G) \leq \Delta(G) + 2$.*

Proof. Obviously, when $\Delta \geq p + 3$ we have $\chi_{p,1}^T(G) \leq C_{p,1}^T(G) \leq \Delta + 2p - 1$ by Theorem 4.1. Since $\chi_{p,1}^T(G) = \lambda_p^T(G) + 1$, let $p = 2$, we have $\lambda_2^T(G) = \chi_{2,1}^T(G) - 1 \leq \Delta + 3 - 1 = \Delta + 2$ with $\Delta \geq 5$. ■

In [3], the author showed that there existed infinitely many outerplanar graphs G such that $\chi_{2,1}^T(G) = \Delta + 3$. So we have $C_{2,1}^T(G) = \Delta + 3$ by Theorem 4.1. That is to say, the upper bound in Theorem 4.1 can not be improved when $p = 2$.

Finally, we conjecture that Theorem 4.1 is also true when $\Delta \leq p + 2$.

Conjecture 4.4 *Let G be an outerplanar graph. Then*

$$C_{p,1}^T(G) \leq \Delta + 2p - 1.$$

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